e-content for students

B. Sc.(honours) Part 2 paper 3

Subject:Mathematics

Topic:Properties of continuous function

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Properties of continuous function

Theorem 1 *If $f(x)$ and $\varphi(x)$ are continuous at $x = a$ then*

(i) $f(x) \pm \varphi(x)$ *are continuous at* $x = a$;

(ii) $f(x) \times \varphi(x)$ *is continuous at* $x = a$;

(iii) $\dfrac{f(x)}{\varphi(x)}$ *is continuous at* $x = a$ *provided*

$\varphi(x) \neq 0$ *for* $a - h \leq x \leq a + h$.

Proof. As $f(x)$ and $\varphi(x)$ are continuous at $x = a$,

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} \varphi(x) = \varphi(a).$$

(i) Let $F(x) = f(x) \pm \varphi(x)$; then $F(a) = f(a) \pm \varphi(a)$.

Now $\lim_{x \to a} F(x) = \lim_{x \to a} [f(x) \pm \varphi(x)]$

$= \lim_{x \to a} f(x) \pm \lim_{x \to a} \varphi(x)$, (by theorem on limit)

$= f(a) \pm \varphi(a) = F(a)$.

$\therefore$ $F(x)$ is continuous at $x = a$.

(ii) Let $G(x) = f(x) \times \varphi(x)$; then $G(a) = f(a) \times \varphi(a)$.

Now $\lim_{x \to a} G(x) = \lim_{x \to a} [f(x) \times \varphi(x)]$

$= \lim_{x \to a} f(x) \times \lim_{x \to a} \varphi(x)$. {by theorem on limit}

$= f(a) \times \varphi(a) = G(a)$.

$\therefore$ $G(x)$ is continuous at $x = a$.

(iii) Let $G(x) = \dfrac{f(x)}{\varphi(x)}$; then $G(a) = \dfrac{f(a)}{\varphi(a)}$.

Not $\lim_{x \to a} G(x) = \lim_{x \to a} \dfrac{f(x)}{\varphi(x)}$

$= \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} \varphi(x)}$, {by theorem on limit}

$= \dfrac{f(a)}{\varphi(a)} = G(a)$.

$\therefore$ $G(x)$ is continuous at $x = a$.

Theorem 2 *If f is continuous, then so is* $|f|$.

Proof : Let f be continuous at a point $x = a \in I$.

Then given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

But we know that

$$||f(x)| - |f(a)|| < |f(x) - f(a)|,$$

that is, $$|f|(x) - |f|(a) < |f(x) - f(a)|$$

Hence combining (1) and (2), we find that

$$|x - a| < \delta \Rightarrow ||f|(x) - |f|(a)| < \varepsilon.$$

This shows that $|f|$ is continuous at $x = a$.

Theorem 3 *If $f(x)$ be continuous in the closed interval* $[a, b]$, *then, given* ϵ, *the interval can always be divided up into a finite number of sub-intervals such that* $|f(x_1) - f(x_2)| < \epsilon$, *where* x_1 *and* x_2 *are any two points in the same sub-interval.*

Proof. Suppose that the theorem is not true.

Let c be the mid-point of $[a, b]$. Then $[a, b]$ is divided into two equal sub-intervals $[a, c]$ and $[c, b]$.

The theorem must not be true in at least one of the two sub-intervals $[a, c]$ and $[c, b]$.

Suppose it is not true in $[c, b]$. Denote this sub-interval by $[a_1, b_1]$. It is evident that the interval $[a_1, b_1]$ lies wholly inside $[a, b]$ and is of length $b_1 - a_1$, that is, $\frac{1}{2}(b - a)$.

Again divide $[a_1, b_1]$ into two equal sub-intervals. We denote the interval in which the theorem is not true by $[a_2, b_2]$. Obviously the sub-interval $[a_2, b_2]$ lies wholly inside $[a_1, b_1]$ and is of length $b_2 - a_2$, that is, $\frac{1}{2}(b_1 - a_1)$, that is,

$$\tfrac{1}{2}\cdot\tfrac{1}{2}(b - a) \Rightarrow \frac{1}{2^2}(b - a).$$

Apply this process of repeated bisection. In this way we get an interval $[a_n, b_n]$ in which the theorem is not true and this interval lies wholly inside the preceding interval $[a_{n-1}, b_{n-1}]$ and is of length $b_n - a_n$, that is, $\frac{1}{2^n}(b - a)$.

$$\therefore \quad \lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{b - a}{2^n} = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = x_0 \text{ (say)}.$$

Suppose, for definiteness, that x_0 does not coincide with a or b.

Since $f(x)$ is continuous at $x = x_0$, therefore, by definition of continuity, there exists a value of δ such that

$$| f(x) - f(x_0) | < \frac{\epsilon}{2}, \text{ when } | x - x_0 | < \delta. \quad \ldots (1)$$

If n be chosen so large that $b_n - a_n$ is less than δ, then the interval $[a_n, b_n]$ is contained entirely within the interval

$$[x_0 - \delta, x_0 + \delta].$$

Let x_1 and x_2 be any two points in (a_n, b_n), then from (1), we get

$$| f(x_1) - f(x_0) | < \frac{\epsilon}{2}$$

and $$| f(x_2) - f(x_0) | < \frac{\epsilon}{2}.$$

Now $f(x_1) - f(x_2) = f(x_1) - f(x_0) + f(x_0) - f(x_2)$

$$\Rightarrow \quad | f(x_1) - f(x_2) | = | \{f(x_1) - f(x_0)\} + \{f(x_0) - f(x_2)\} |$$

$$\Rightarrow \quad | f(x_1) - f(x_2) | < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\Rightarrow \quad | f(x_1) - f(x_2) | < \epsilon.$$

This is a contradiction to our supposition. Hence our supposition is wrong. In other words, the theorem must be true.

Theorem 4:prove that a function w
hich is continuous in a closed inter
val[a b] is bounded therein

Proof. We know that if $f(x)$ be continuous in the closed interval $[a, b]$, then, given ϵ, the interval can always be divided up into a finite number of sub-intervals such that

$$|f(x_1) - f(x_2)| < \epsilon, \quad \dots (1)$$

where x_1 and x_2 are any two points in the same sub-interval.

Let the dividing points be $x_0 = a, x_1, x_2, \dots, x_{n-1}, x_n = b$.

Let x be any point in the first sub-interval $[a, x_1]$.

Then, from (1), we have

$$|f(a) - f(x)| < \epsilon. \quad \dots (2)$$

Now $f(x) = f(a) + \{f(x) - f(a)\}$

$\Rightarrow \quad |f(x)| = |f(a) + \{f(x) - f(a)\}|$

$\Rightarrow \quad |f(x)| \leqslant |f(a)| + |f(x) - f(a)|$

$\Rightarrow \quad |f(x)| < |f(a)| + \epsilon$, using (2).

In particular, when $x = x_1$,

$$|f(x_1)| < |f(a)| + \epsilon. \quad \dots (3)$$

Again, let x be any point in the second sub-interval $[x_1, x_2]$.

Then from (1), we have

$$|f(x_1) - f(x)| < \epsilon. \quad \dots (4)$$

Now $f(x) = f(x_1) + \{f(x) - f(x_1)\}$

$\Rightarrow \quad |f(x)| = |f(x_1) + \{f(x) - f(x_1)\}|$

$\Rightarrow \quad |f(x)| \leqslant |f(x_1)| + |f(x) - f(x_1)|$

$\Rightarrow \quad |f(x)| < |f(x_1)| + \epsilon$, from (4)

$\Rightarrow \quad |f(x)| < |f(a)| + 2\epsilon$, using (3).

In particular, when $x = x_2$,

$$|f(x_2)| < |f(a)| + 2\epsilon.$$

By proceeding in this way we get, when x is any point in n^{th} sub-interval $[x_{n-1}, b]$,

$$|f(x)| < |f(a)| + n\epsilon.$$

This inequality is true for the whole interval $[a, b]$, that is, all the values of $f(x)$ in the interval $[a, b]$ lie between $f(a) - n\epsilon$ and $f(a) + n\epsilon$.

Hence $f(x)$ is bounded in $[a, b]$.